Level sets of smooth Gaussian fields

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Smooth Gaussian fields

- Let $V \in \mathbb{R}^d$ be an open set. A C^k*-smooth Gaussian field* is a Gaussian process indexed by V which has C^k-smooth sample paths. [Fields are centered throughout this talk]
- **Kolmogorov theorem**: Suppose that $K : V \times V \to \mathbb{R}$ is a positive definite symmetric function of class $C^{k,k}(V \times V)$ and, in addition, that

$$N := \max_{|\alpha|,|\beta| \le k} \sup_{x,y \in V} |\partial_x^\alpha \partial_y^\beta K(x,y)| < \infty.$$

Then there exists a (unique up to an equivalence of distribution) C^{k-1} Gaussian function f on V with the covariance kernel K. Moreover, $\mathbb{E}||f||_{C^{k-1}} \le C\sqrt{N}$.

Stationary fields

- Call a Gaussian field on $\mathbb{R}^d$ *stationary* or *translation invariant* if its covariance kernel $K(x,y)$ depends only on $x-y$, say $K(x,y)=k(x-y)$.
- **Bochner theorem**: For a continuous k, k is a Fourier transform of a finite symmetric ($\rho(A)=\rho(-A)$) positive Borel measure ρ on $\mathbb{R}^d$, i.e.

$$k(x)=\int_{\mathbb{R}^d} e^{2\pi i(\lambda\cdot x)}d\rho(\lambda).$$

- Call ρ the *spectral measure* of the field.
- The field is a fourier transform of white noise on ρ, i.e.

$$f(x)=W_\rho(e^{2\pi i x\cdot t})$$

The properties of f, K, ρ are closely related.

Examples: Random plane waves

- Spectral measure is (normalised) arc length measure on $S^1 \subset \mathbb{R}^2$. So covariance kernel is $J_0(|x-y|)$, where J_0 is zeroth Bessel function.
- Local scaling limit of a number of other Gaussian fields. E.g. Random spherical harmonics [Wig22].
- In 1977, M. Berry [Ber77] conjectured that the wavefunctions of generic chaotic domains has same statistical behaviour as that of RPW in the high energy limit. Later, he gave precise characterisation of chaotic behaviour in terms of length nodal lines of the wavefunctions.

Examples: Bargmann-Fock field

- Covariance kernel is $K(x, y) = e^{-|x-y|^2/2}$. Hence the spectral measure has Gaussian-type density.
- The field can be written as

$$f(x) = e^{-|x|^2/2} \sum_{n,m\geq 0} \frac{a_{n,m}}{\sqrt{n!m!}} x_1^n x_2^m$$

- Thought of as a limit of Gaussian ensemble of homogeneous polynomials. So zero sets are "portrait of 'typical' algebraic variety".
- Many percolation theoretic properties are easier to establish in this model because the correlation decay is very fast.

What we're interested in

- Geometry of level sets of the field f (i.e. the set $\{x : f(x) = l\}$, for some $l \in \mathbb{R}$). Called *nodal set* for the case $l = 0$.
- The Hausdorff measure of the nodal sets. Say, length of nodal lines of planar fields.
- Specifically, estimating the (difference in) measures of level sets of *coupled* fields.

Some of the other well studied topics include percolation theoretic properties of nodal sets, nodal domain counts etc.

So far...

- Distribution of local quantities (e.g. critical points) are well studied (Integral formulas such as Kac-Rice).
- E.g. 1. CLT for level functionals of stationary Gaussian fields using Wiener chaos expansion and Kac-Rice formula [KL01].
- E.g. 2. Asymptotics of variation of nodal lengths in airthmetic random waves cases. [KKW12]

But when comparing geometric observable of two coupled fields, Kac-Rice formulas might not help much.

Kac-Rice formula

- **Kac-Rice formula** (for length of level lines): Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth Gaussian field (with some minimal regularity). Let $U \subset \mathbb{R}^2$ be a Borel set. Then,

$$\mathbb{E}[L(a, U)] = \int_U \mathbb{E}[|\nabla f(x)||f(x) = a]p_{f(x)}(a)dx$$

where $L(a, U)$ is the length of $\{f = a\}$ in U, p_X is the pdf of random variable X.

- Loads of variants of this formula are available. For example, second moment is an integral of two-point correlation function of the field.
- Think of a stochastic version of the co-area formula.

Main result- setup

Fix a domain $D = [-R, R]^d \subset \mathbb{R}^d$. Consider two (coupled) C^2-smooth stationary Gaussian fields $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$ such that the field $F = f_1 - f_2$ has the C^2-fluctuations

$$\sigma_D^2 := \sup_{x \in D} \sup_{|\alpha| \leq 2} \operatorname{Var}[\partial^\alpha F(x)].$$

Let $\mathcal{L}^n$ denote n-dimensional Lebesgue measure. Let $\mathcal{H}^n$ denote the n- dimensional Hausdorff measure.

Main result

Theorem (Beliaev-H. 2023) [BH23, Theorem 2.2]

Let $\mathcal{H}^{d-1}(f_i^{-1}(a))$ denote the measure of level sets in the domain D. Under mild non-degeneracy conditions on the fields, we have

$$\mathbb{E}|\mathcal{H}^{d-1}(f_1^{-1}(0)) - \mathcal{H}^{d-1}(f_2^{-1}(0))| \leq C(f_1, f_2)(\mathcal{L}^d(D)\sqrt{\log R})\sigma_D^{1/7}$$

assuming σ_D is small enough (say, $\sigma_D < 1$). Here, the constant $C(f_1, f_2)$ depends only on the laws of the fields and not the coupling.

Remarks

- The exponent $1/7$ in the RHS of the estimate is not optimal (1 seems to be optimal).
- The factor $\sqrt{\log R}$ is from the quantitative version of Kolmogorov's existence theorem.
- Many schemes for coupling stationary Gaussian fields.

As a corollary of a key lemma in the proof, we can compute average mean curvature of level sets explicitly. See [BH23, Corollary 2.4]. We have,

$$\mathbb{E}[\kappa|f = a] = -a\mathbb{E}|\nabla f|.$$

Proof sketch (1)

- We derive the following equation using Green's formula,

$$\mathcal{H}^{d-1}(f^{-1}(b)) - \mathcal{H}^{d-1}(f^{-1}(a)) = \underbrace{\iint_D \kappa \mathbb{1}_{f\in[a,b]} \mathrm{dVol}}_{\text{Bulk term}} - \underbrace{\oint_{\partial D} \left\langle \frac{\nabla f}{|\nabla f|}, \hat{n} \right\rangle \mathbb{1}_{f\in[a,b]} \mathrm{dS}}_{\text{Boundary term}} . \qquad (*) \tag{1}$$

- We used the following expression for curvature from classical differential geometry,

$$\kappa = \frac{1}{d-1} \mathrm{div} \left(\frac{\nabla f}{|\nabla f|} \right) .$$

Proof sketch (2)

- Now take the difference of the equation $(*)$ for the two fields and evaluate it at $b = 0, a = -\infty$.
- We bound the bulk and boundary term separately.
- Other terms are dominated by

$$\mathbb{E}\left[\left|\int_D (\kappa_1 - \kappa_2)\mathbb{1}[f_1, f_2 < 0] d\mathrm{Vol}\right|\right]$$

  when σ_D is small enough.
- Bound expectation of $|\kappa_1 - \kappa_2|$ when $|\nabla f_i|$ are not too small. (Pointwise estimate)

What next?

- Prove it in manifold setting. Important because of mathematical physics motivation.
- Similar estimates for higher moments.
- Extension for non-stationary fields.

A potential application [Work in progress]

An alternate proof of CLT for measure of level sets of stationary Gaussian fields.

- Using a spatial verison of martingale CLT (see [Pen01], [BMM22]), we can give a more geometric proof.
- More straightforward proof than Wiener choas or fourth moment method.

End

Thank you!

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